Generic Multi-scale Segmentation and Curve Approximation Method

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Abstract. We propose a new complete method to extract significant description(s) of planar curves according to constant curvature segments. This method is based *(i)* on a multi-scale segmentation and curve approximation algorithm, defined by two grouping processes (polygonal and constant curvature approximations), leading to a multi-scale covering of the curve, and *(ii)* on an intra- and inter-scale classification of this multi-scale covering guided by heuristically-defined qualitative labels leading to pairs (scale, list of constant curvature segments) that best describe the shape of the curve. Experiments show that the proposed method is able to provide salient segmentation and approximation results which respect shape description and recognition criteria.

1 Introduction

In order to easily manipulate a planar curve or databases composed of planar curves, it would be interesting to represent data according to primitives which describe them in a way that respects their actual shape for recognition and compression purposes. In this paper, we present an improved version of the multi-scale segmentation and curve approximation method introduced in [1][6]. It is related to the category of the methods favoring shape *recovery* [2][3][8]. However, this new method tries to go behind limitations generated by the other methods by identifying in a formal way the requirements related to the problem of segmentation and approximation of a planar curve [1][7]. The associated algorithm represents a generalization of the paradigm *recover-and-select* established by Leonardis and Bajcsy [4].

The original method that we propose in order to extract significant description(s) of planar curves into lists of constant curvature segments (such as straight line segments and/or circular arcs) is based on MuscaGrip (which stands for *MUlti-scale Segmentation and Curve Approximation based on the Geometry of Regular Inscribed Polygons*), a multi-scale segmentation and curve approximation algorithm, leading to a multi-scale covering of the curve. The MuscaGrip algorithm is defined by two grouping processes: *(i)* a polygonal approximation (from points to straight line segments), and *(ii)* a constant curvature approximation (from straight line segments to straight line segments and/or circular arcs).

MuscaGrip repeats the first grouping process using each point on the curve as its starting point and the second grouping process using each straight line segment provided by the first process as its starting segment. These repetitions lead to a complete description of the curve composed of lists of constant curvature segments at different scales. Although they increase the computational load of the algorithm, these repetitions are necessary in order to respect invariance criteria. In order to find a set of pairs composed of one scale and one list of constant curvature segments that best describe the shape of the curve, a global combinatorial method of the multi-scale covering is introduced, guided by heuristically-defined qualitative labels leading to a single non-redundant subset.

In the following, the description of the complete method is presented. The MuscaGrip grouping processes are first described. The method to extract the minimal set of adequate pairs (scale, list of constant curvature segments) is then introduced in details with its inter- and intra-scale classification steps. Finally, experimental results are presented for open and closed planar curves.

2 MuscaGrip: Point Grouping Process

The generic process first splits a planar curve into several sub-curves, each of which is approximated by a straight line segment. The associated point grouping criterion is equivalent to a *co-circularity* criterion among the connected points of the sub-curve. A *scale parameter*, acting as a maximum deviation criterion, is associated with a scale measure.

It is assumed that a point chain of two points forms an uniform sub-curve. A chain of three or more points forms an uniform sub-curve if and only if the perpendicular distance of each point of the chain relative to the straight line joining the two endpoints of the chain is less than or equal to the scale parameter. The computation of this step is repeated at a number of scales to provide a multi-scale set of polygonal approximations, and using all points on the curve as a starting point.

More formally, let $\mathcal{C}$ be an open or closed planar curve, an ordered list of n points $p_i(x_i, y_i)$, for $i \in [1, n]$, where p_i, p_{i+1} are consecutive points along sampled curve, then $\mathcal{C} = \{\, p_i(x_i, y_i) \,|\, i \in [1, n] \wedge x_i \in \mathcal{R} \wedge y_i \in \mathcal{R}\,\}$. When $\mathcal{C}$ is open, p_1 is obtained from the first point of the curve and p_n from the last point. Otherwise, p_1 (consequently p_n) is obtained from an arbitrarily selected point of $\mathcal{C}$. Let $\mathcal{S}$ be an ordered list of m scales s_j, for $j \in [1, m]$, where s_j, s_{j+1} are consecutive scales, then $\mathcal{S} = \{\, s_j \,|\, j \in [1, m] \wedge s_j \in \mathcal{R}\}$ with s_1 the finest scale, and s_m the coarsest scale.

At a given scale $s_j \in \mathcal{S}$, $PA_{\mathcal{C}}(s_j, p_i)$, a polygonal approximation associated to a planar curve $\mathcal{C}$ and generated from point $p_i \in \mathcal{C}$ is defined by an ordered list of p straight line segments $sls_k(p_q, p_r)$, for $k \in [1, p]$, whose p_q and p_r are the first and last points of an uniform sub-curve of $\mathcal{C}$. We thus have:

$$PA_{\mathcal{C}}(s_j, p_i) = \{\, sls_k(p_q, p_r) \,|\, k \in [1, p] \wedge p_q, p_r \in \mathcal{C}\,\}. \tag{1}$$

For a closed planar curve, $PA_{\mathcal{C}}(s_j, p_i)$ is defined by $PA_{\mathcal{C}}^{clkw}(s_j, p_i)$ in the clockwise direction and $PA_{\mathcal{C}}^{cclkw}(s_j, p_i)$ in the counter-clockwise direction (overshoot can occur).

3 MuscaGrip: Straight Line Segment Grouping Process

For a polygonal approximation, the constant curvature approximation process aims at grouping $n \geq 2$ adjacent straight line segments into circular arcs whenever feasible. The associated uniformity criterion is based on the model of a *regular polygon*, formed of $n \geq 2$ segments, approximating the circular arc (noted *ca* in most figures) into which it is *inscribed*. Let a be the radius of the inscribed circular arc, and let R be the radius of the circumscribed circular arc, the difference between R and a is related to s_j, the scale parameter. The constant curvature approximation is then obtained using a merging process of consecutive straight line segments of the polygonal approximation.

Given a polygonal approximation and a regular polygon whose features are induced by a sublist of this polygonal approximation, is it possible that, by adding to this sublist a straight line segment being adjacent to it, the new sublist still be at the basis of a regular polygon whose features are similar to those of the old instance? If such is the case after consideration of a set of uniformity criteria, a new straight line segment adjacent to the sublist is targeted whenever possible.

More formally, let $\mathcal{P}'$ be a sublist of a polygonal approximation composed of p straight line segments. $\mathcal{P}'$ is defined by an ordered list of $\wp'$ segments, with $\wp' \leq p$. If $\wp'$ is equal to 1, $\mathcal{P}'$ is only composed of one straight line segment, then $\mathcal{P}'$ is uniform. Before continuing, let us note that a regular polygon $\mathcal{RP}'$ originating from an uniform sublist $\mathcal{P}'$ is entirely described by *(i)* the angle θ' between two consecutive sides, *(ii)* the length l' of each side and *(iii)* n', its number of sides. Derived features are deduced, *(i)* R', the value of the radius of the circumscribed circular arc, and *(ii)* a', the value of the radius of the inscribed circular arc, commonly called apothem. If $\wp'$ is equal to 2, $\mathcal{P}'$ composed by two straight line segments, sls_1 and sls_2, will be considered uniform if and only if the regular polygon $\mathcal{RP}'$ originating from this one is validated by the following uniformity criteria:

1• $s_j - \delta s \leq R' - a'$ and $R' - a' \leq s_j + \delta s$. δs corresponds to the step between two consecutive scales of $\mathcal{S}$,

2• the features (l_{sls_1} and l_{sls_2}, the lengths of the segments, and $\theta_{(sls_1, sls_2)}$, the angle between the two segments) describing $\mathcal{P}'$ must be validated by the features describing two regular control polygons, $\mathcal{RP}'_{s_j - \delta s}$ and $\mathcal{RP}'_{s_j + \delta s}$. These latter are induced by R' and the scales $s_j - \delta s$ and $s_j + \delta s$.

When $\mathcal{P}'$ is uniform, a first instance of a regular polygon $\mathcal{RP}'$ inscribed into the approximating circular arc is created.

If $\mathcal{P}'$ is uniform then the sublist $\mathcal{P}$ composed of $\wp$ ($\wp = \wp' + 1$) straight line segments, and defined by $\mathcal{P} = \{\, \mathcal{P}' \cup \{sls\} \,|\, sls \in PA \,\}$ is also uniform if and only if a regular polygon $\mathcal{RP}$, defined by θ, l and n, can be deduced from $\mathcal{P}$ and $\mathcal{RP}'$ according to various uniformity criteria:

1• $\wp$ is equal to or higher than 2,

2• $\wp$ is lower than or equal to n', the number of sides of $\mathcal{RP}'$, the regular polygon induced by $\mathcal{P}'$,

3• $s_j - \delta s \leq R - a$ and $R - a \leq s_j + \delta s$, 4• the features (length of the segments, angle between two consecutive segments) describing $\mathcal{P}$ must be validated by the features describing two regular control polygons, $\mathcal{RP}_{s_j-\delta s}$ and $\mathcal{RP}_{s_j+\delta s}$. These latter are induced by R and the scales $s_j - \delta s$ and $s_j + \delta s$.

At a given scale $s_j \in \mathcal{S}$, $CCA_{\mathcal{C}}(s_j, p_i, sls_k)$, a constant curvature approximation related to a polygonal approximation $PA_{\mathcal{C}}(s_j, p_i)$ and initiated by the straight line segment sls_k is defined by an ordered list of $q+r$ constant curvature segments ccs_s for $s \in [1, q+r]$: q straight line segments sls_u for $u \in [1, q]$ and r circular arcs ca_v for $v \in [1, r]$. ca_v is provided by the grouping of an ordered list of straight line segments according to the uniformity criteria listed above. We thus have:

$$CCA_{\mathcal{C}}(s_j, p_i, sls_k) = \{\, ccs_s \mid s \in [1, q+r] \wedge (ccs_s \in PA_{\mathcal{C}}(s_j, p_i) \vee ccs_s = \cup sls) \,\}. \quad (2)$$

Overlap can occur inside $CCA_{\mathcal{C}}(s_j, p_i, sls_k)$. Once again, this step is repeated using all straight line segments provided by the polygonal approximation as a starting segment.

4 Extraction of The Best Descriptions

A significant computational load results from the proposed multi-scale segmentation and approximation of a planar curve $\mathcal{C}$. This multi-scale method leads to many representations. Among them, only the more salient ones should be considered. For that purpose, we define an intra- and inter-scales classification of this multi-scale description, guided by heuristically-defined qualitative labels leading to a set of representation(s) which respect shape description and recognition criteria.

4.1 Labeling of Polygonal Approximations

A classification of the results obtained from the first grouping process is a good starting point for extracting salient approximations. We associate a *qualitative label* to each polygonal approximation associated with both open and closed curves. Three labels are defined, label VG_{PA} for *Very Good Polygonal Approximation*, label G_{PA} for *Good Polygonal Approximation*, and label A_{PA} for *Acceptable Polygonal Approximation*.

In the case of an open curve $\mathcal{C}$, at scale s_j, *(i)* label VG_{PA} means that endpoints of $\mathcal{C}$, p_1 and p_n, are *real* endpoints of $PA_{\mathcal{C}}(s_j, p_i)$, *(ii)* label G_{PA} means that p_1 or p_n is a *virtual* endpoint of $PA_{\mathcal{C}}(s_j, p_i)$, the other being real, and *(iii)* label A_{PA} means that p_1 and p_n are both virtual endpoints of $PA_{\mathcal{C}}(s_j, p_i)$. In the same way, for a closed curve $\mathcal{C}$, at scale s_j, *(i)* label VG_{PA} means that p_i is the starting and ending point of $PA_{\mathcal{C}}^{clkw}(s_j, p_i)$ and $PA_{\mathcal{C}}^{cclkw}(s_j, p_i)$, *(ii)* label G_{PA} means that p_i is the starting and ending point of $PA_{\mathcal{C}}^{clkw}(s_j, p_i)$ or $PA_{\mathcal{C}}^{cclkw}(s_j, p_i)$, and *(iii)* label A_{PA} means that p_i is the starting and ending point of neither $PA_{\mathcal{C}}^{clkw}(s_j, p_i)$ and $PA_{\mathcal{C}}^{cclkw}(s_j, p_i)$. In the latter two cases, overshoot occurs. If

$$\begin{aligned} \mathcal{S}(VG_{PA_{\mathcal{C}}}) &= \{PA_{\mathcal{C}}(s_j, p_i) \mid label = VG_{PA}\}, \\ \mathcal{S}(G_{PA_{\mathcal{C}}}) &= \{PA_{\mathcal{C}}(s_j, p_i) \mid label = G_{PA}\}, \\ \mathcal{S}(A_{PA_{\mathcal{C}}}) &= \{PA_{\mathcal{C}}(s_j, p_i) \mid label = A_{PA}\}, \end{aligned} \quad (3)$$

where $\mathcal{S}(X_{PA_{\mathcal{C}}})$ is a set composed of polygonal approximations of $\mathcal{C}$ labeled X, then,

$$\begin{aligned} \mathcal{S}(VG_{PA_{\mathcal{C}}}) \cap \mathcal{S}(G_{PA_{\mathcal{C}}}) &= \emptyset \\ \mathcal{S}(G_{PA_{\mathcal{C}}}) \cap \mathcal{S}(A_{PA_{\mathcal{C}}}) &= \emptyset \\ \mathcal{S}(VG_{PA_{\mathcal{C}}}) \cap \mathcal{S}(A_{PA_{\mathcal{C}}}) &= \emptyset. \end{aligned} \tag{4}$$

Therefore,

$$\mathcal{S}(VG_{PA_{\mathcal{C}}}) \cup \mathcal{S}(G_{PA_{\mathcal{C}}}) \cup \mathcal{S}(A_{PA_{\mathcal{C}}}) = PA_{\mathcal{C}}(s_j). \tag{5}$$

4.2 Labeling of Constant Curvature Approximations

Following the first grouping process, a labeled polygonal approximation $PA_{\mathcal{C}}(s_j, p_i)$ leads to $CCA_{\mathcal{C}}(s_j, p_i)$ composed of p $CCA_{\mathcal{C}}(s_j, p_i, sls_k)$. Each one can in turn be qualitatively labeled: label VG_{CCA} for *Very Good Constant Curvature Approximation*, label G_{CCA} for *Good Constant Curvature Approximation*, and label A_{CCA} for *Acceptable Constant Curvature Approximation*.

For both open and closed curves, *(i)* label A_{CCA} means that overlap (and overshoot if $\mathcal{C}$ is closed) occurs into $CCA_{\mathcal{C}}$ (s_j, p_i, sls_k), *(ii)* label G_{CCA} means that no overlap (and no overshoot if $\mathcal{C}$ is closed) occurs into $CCA_{\mathcal{C}}(s_j, p_i, sls_k)$ but the ratio between the number of constant curvature segments and the number of straight line segments from $PA_{\mathcal{C}}$ (s_j, p_i) is close to 1.0, consequently $CCA_{\mathcal{C}}(s_j, p_i, sls_k)$ is composed principally of straight line segments and then

$$CCA_{\mathcal{C}}(s_j, p_i, sls_k) \cong PA_{\mathcal{C}}(s_j, p_i), \tag{6}$$

and *(iii)* label VG_{CCA} means that no overlap (and no overshoot if $\mathcal{C}$ is closed) occurs into $CCA_{\mathcal{C}}(s_j, p_i, sls_k)$ and the ratio (also called *compression rate*) between the number of circular arcs and the number of constant curvature segments is high. When s_j is coarse, G_{CCA} is used more often than VG_{CCA} because searching to group adjacent straight line segments into circular arcs is less feasible, the number of straight line segments into $PA_{\mathcal{C}}(s_j, p_i)$ decreasing with increasing scale s_j. If

$$\begin{aligned} \mathcal{S}(VG_{CCA_{\mathcal{C}}}) &= \{CCA_{\mathcal{C}}(s_j, p_i, sls_k) \mid label = VG_{CCA}\}, \\ \mathcal{S}(G_{CCA_{\mathcal{C}}}) &= \{CCA_{\mathcal{C}}(s_j, p_i, sls_k) \mid label = G_{CCA}\}, \\ \mathcal{S}(A_{CCA_{\mathcal{C}}}) &= \{CCA_{\mathcal{C}}(s_j, p_i, sls_k) \mid label = A_{CCA}\}, \end{aligned} \tag{7}$$

where $\mathcal{S}(X_{CCA_{\mathcal{C}}})$ is a set composed of constant curvature approximations of $\mathcal{C}$ labeled X, then

$$\begin{aligned} \mathcal{S}(VG_{CCA_{\mathcal{C}}}) \cap \mathcal{S}(G_{CCA_{\mathcal{C}}}) &= \emptyset \\ \mathcal{S}(G_{CCA_{\mathcal{C}}}) \cap \mathcal{S}(A_{CCA_{\mathcal{C}}}) &= \emptyset \\ \mathcal{S}(VG_{CCA_{\mathcal{C}}}) \cap \mathcal{S}(A_{CCA_{\mathcal{C}}}) &= \emptyset. \end{aligned} \tag{8}$$

Therefore,

$$\mathcal{S}(VG_{CCA_{\mathcal{C}}}) \cup \mathcal{S}(G_{CCA_{\mathcal{C}}}) \cup \mathcal{S}(A_{CCA_{\mathcal{C}}}) = CCA_{\mathcal{C}}(s_j, p_i). \tag{9}$$

The most interesting description for a curve is a set of one or several VG_{CCA} provided by a VG_{PA}. If no VG_{PA} exists, then the most interesting description for a curve is a set of one or several VG_{CCA} provided by a G_{PA}. The compression rate allows to partition $\mathcal{S}(CCA_C)$. If several $CCA_C(s_j, p_i, sls_k)$ have the same compression rate then they form a partition of $\mathcal{S}(CCA_C)$, and they can in turn be classified according to the accumulation of the errors (also called *error rate*) generated between each pair of adjacent constant curvature segments. A good compression rate and a weak error rate are thus significant factors.

5 Results

This section presents results for various open and closed curves. To generate results, the algorithm proceeds as follows: for each curve $\mathcal{C}$, at each scale $s_j \in \mathcal{S}$, search for $PA_{\mathcal{C}}(s_j, p_i)$ labeled VG_{PA}, then search by intra-and inter-scales classification for the most significant $CCA_{\mathcal{C}}(s_j, p_i,\ sls_k)$ labeled VG_{CCA}.

Fig.1(a) provides for a spiral of Archimede-shaped open curve $\mathcal{C}$ the best constant curvature approximation hypothesis for one scale. In order to highlight the span of the circular arcs, grey lines are drawn. Best description hypothesis for a semi-limacon of Pascal-shaped open curve is shown in Fig.1(b) for working scales $s_j \in [1.0, 3.0]$ with $\delta s = 1.0$. For these two results, we can appreciate the excellent compression rate of data.

Invariance to similarity transformations such as translation, rotation and scaling is an important criterion to which a good algorithm of segmentation and approximation of planar curves must conform to in order to provide similar descriptions under various conditions. In order to show invariance, four different orientations are used on an ellipse-shaped closed curve and results are shown on Fig.2. For this curve and under any condition, each obtained description, formed by four circular arcs, is representative of the geometrical shape. Let us note that the origin of each circular arc is located on the axes of symmetry of the curves. In order to visualize the behavior of the algorithm more adequately in the presence of the same curve at several scales, we chose to show results on one set made up of astroids. Whatever the scale to which the curve appears, its general description must remain the same. The results shown on Fig.3(a) illustrate the very good behavior of the algorithm relative to scaling.

An interesting aspect of the MuscaGrip algorithm is the conservation of existing symmetries. Fig.3(b) illustrates this fact by experimenting on a rose-shaped closed curve whose complexity is high. The rose is formed by ten petals and the constant curvature approximation hypothesis is particularly convincing. Each petal is described in the same way. Only circular arcs are included in the description. The conservation of symmetries is also visible on Fig.2 and Fig.3(a). A polygonal approximation of recursive subdivision type, [5], can reduce the overall process only when a planar curve is composed of symmetries. On the other hand, in the case of an unspecified curve, a polygonal approximation as recommended in MuscaGrip is necessary and impossible to circumvent.

When noise is added along the curve, the method has to find a final description including the same number of primitives and the same type of primitives as description that one would have obtained without noise. In the presence of noise, it is obvious that a multi-scale method is more suitable and more robust because no matter what occurs, there always exists one scale likely to attenuate it. Results shown on Fig.4 for an ellipse-shaped closed curve are very satisfactory because, in spite of the more or less significant irregularity in the signal, the algorithm is able to provide an acceptable description.

6 Conclusion

A complete method to extract significant descriptions of planar curves as ordered lists of constant curvature segments was presented. This method is based *(i)* on MuscaGrip, a multi-scale segmentation and curve approximation algorithm, defined by two grouping processes leading to a multi-scale covering of the curve, and *(ii)* on an intra- and inter-scale classification of this multi-scale covering, guided by qualitative labels, leading to a single non-redundant subset. The goal is to find a minimal set of pairs composed of *(scale,ordered list of constant curvature segments)* to best describe the shape of the curve. Experiments on synthetic curves have shown that the proposed method is able to provide salient segmentation and approximation results which respect shape description and recognition criteria, and which have a good data compression rate. A more exhaustive experimental evaluation of algorithms on curves of various types, ideal and noisy, and contours from real $2D$ illuminance images is presented in [7] and confirm the good behavior of the method. Furthermore, this research work is part of a more generic project for detecting and describing 3D objects in a single 2D image based on high-level structures obtained by perceptual grouping of constant curvature segments [7].

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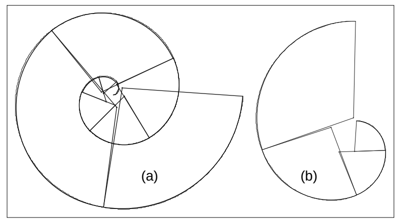


Fig. 1. Best constant curvature approximation hypothesis. (a) For a spiral of Archimede-shaped open curve $\mathcal{C}$ composed of 1440 points, at scale $s_j = 1.0$ with $\delta s = 0.15$, for $PA_{\mathcal{C}}(s_j, p_i)$ $(= 45sls)$ labeled VG_{PA}, $CCA_{\mathcal{C}}(s_j, p_i, sls_k) = (3sls, 8ca)$. (b) For a semi-limacon of Pascal-shaped open curve $\mathcal{C}$ composed of 360 points, at scales $s_j \in [1.0, 3.0]$ with $\delta s = 1.0$, for $PA_{\mathcal{C}}(s_j, p_i)$ labeled VG_{PA}, $CCA_{\mathcal{C}}(s_j, p_i, sls_k) = (0sls, 4ca)$).

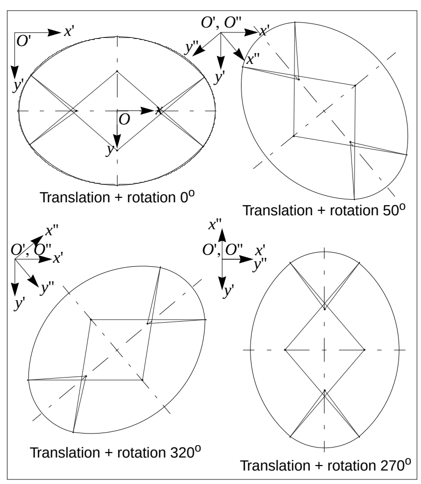
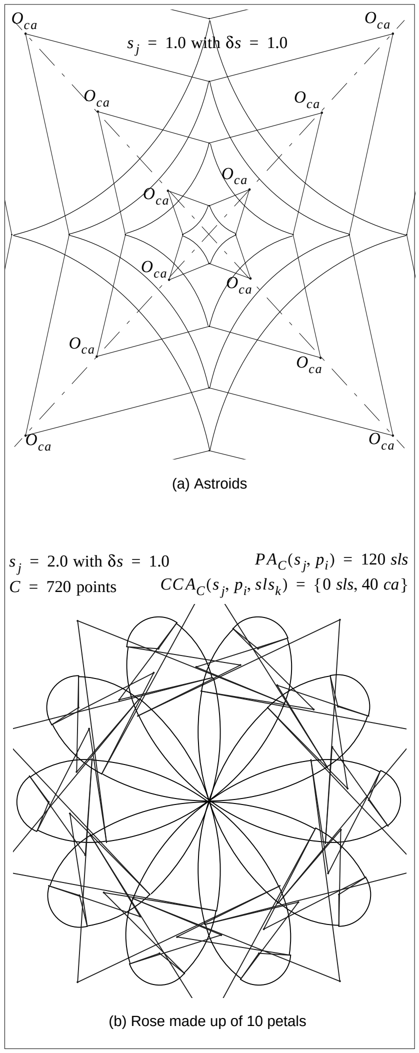


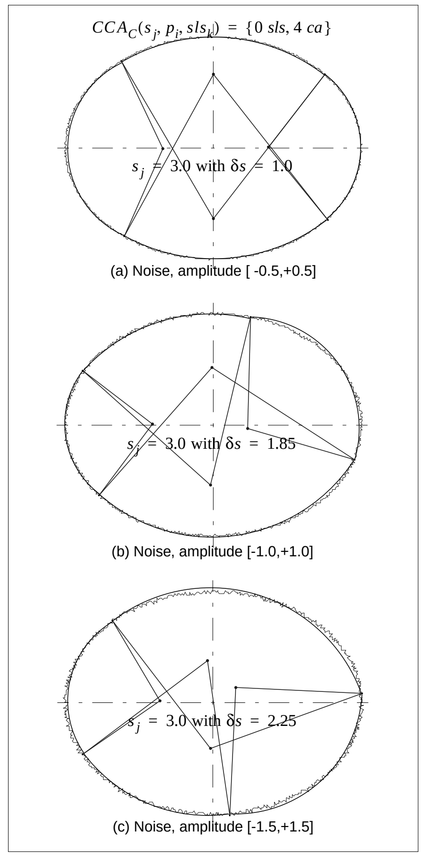
Fig. 2. Invariance to translation and rotation for an ellipse-shaped closed curve, composed of 720 points, at scale $s_j = 3.0$.



(a) Astroids

(b) Rose made up of 10 petals

Fig. 3. (a) Invariance to scaling for a set of closed curves: astroids. (b) Conservation of existing symmetries on a closed curve: a rose made up of ten petals.



(a) Noise, amplitude [-0.5,+0.5]

(b) Noise, amplitude [-1.0,+1.0]

(c) Noise, amplitude [-1.5,+1.5]

Fig. 4. Behavior face to adding noise along the curve.